

An N-State Endogenous Markov-Switching Model with Applications in Macroeconomics and Finance *

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Abstract: We develop an N -regime Markov-switching model in which the latent state variable driving the regime switching is endogenously determined with the model disturbance term. The model's structure captures a wide variety of patterns of endogeneity, and yields a simple test of the null hypothesis of exogenous switching. We derive an iterative filter that generates objects of interest, including the model likelihood function and estimated regime probabilities. Using simulation experiments, we demonstrate that the maximum likelihood estimator performs well in finite samples and that a likelihood ratio test of exogenous switching has good size and power properties. We provide results from two applications of the endogenous switching model: a three-state model of U.S. business cycle dynamics and a three-state volatility model of U.S. equity returns. In both cases we find statistically significant evidence in favor of endogenous switching.

Keywords: Regime-switching, business cycle asymmetry, nonlinear models, volatility feedback, equity premium

JEL Classifications: C13, C22, E32, G12

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1 Introduction

Regression models with time-varying parameters have become a staple of the applied econometrician’s toolkit. A particularly prevalent version of these models is the Markov-switching regression of Goldfeld and Quandt (1973), in which parameters switch between some finite number of regimes, and this switching is governed by an unobserved Markov process. Hamilton (1989) makes an important advance by extending the Markov-switching framework to an autoregressive process, and providing an iterative filter that produces both the model likelihood function and filtered regime probabilities. Hamilton’s paper initiated a large number of applications of Markov-switching models, and these models are now a standard approach to describe the dynamics of many macroeconomic and financial time series. Hamilton (2008) and Piger (2009) provide surveys of this literature.

Hamilton’s Markov-switching regression model assumes that the Markov state variable governing the timing of regime switches is strictly exogenous, and thus independent of the regression disturbance at all leads and lags. Diebold et al. (1994) and Filardo (1994) extend the Hamilton model to allow the transition probabilities governing the Markov process to be partly determined by strictly exogenous or predetermined information, which could include lagged values of the dependent variable. However, this time-varying transition probability (TVTP) formulation maintains the assumption that the state variable is independent of the contemporaneous value of the regression disturbance. The large literature applying Markov-switching models has almost exclusively focused on either the Hamilton (1989) fixed transition probability model or the TVTP extension, which we will collectively refer to as “exogenous switching” models.

Despite the popularity of this exogenous switching framework, it is natural in many applications to think of the state process as contemporaneously correlated with the regression disturbance, which we refer to as “endogenous switching”. For example, a common application of the Markov-switching regression is to models where the dependent variable is an aggregate measure of some macroeconomic or financial variable, and the state variable

is meant to capture the business cycle regime (e.g. expansion and recession). It seems reasonable that shocks to these aggregate quantities, such as real GDP, would contribute simultaneously to changes in the business cycle phase. More generally, both the state variable and the disturbance term to the dependent variable may be influenced simultaneously by a number of unmodeled elements. For example, in the Hamilton (1989) regime-switching autoregressive model of real GDP growth, both the state variable capturing the business cycle phase and the shock to real GDP are likely influenced by other factors, such as monetary and fiscal policy.

Motivated by such arguments, Kim et al. (2008) develop an endogenous switching regression model, in which the state variable and the regression disturbance term are determined simultaneously. Kang (2014) incorporates the Kim et al. (2008) model of endogenous switching inside of a more general state-space model. However, a significant drawback of this existing endogenous switching literature is that it is largely limited to the case of two regimes.¹ This limits the potential application of the model considerably, as there is evidence for more than two regimes in many empirical implementations of the Markov-switching model. For example, in models of real activity, Boldin (1996) finds evidence for a three regime switching model of business cycle dynamics for real GDP, while Hamilton (2005) does the same for the unemployment rate. For asset prices, Garcia and Perron (1996) and Guidolin and Timmermann (2005) provide evidence for a three-regime switching mean and volatility model of U.S. interest rates and equity returns respectively. In a Markov-switching VAR, Sims and Zha (2006) find the best fit using nine regimes, primarily capturing changes in conditional volatility.

In this paper, we develop an N -regime endogenous Markov-switching regression model. In the two regime case, the model collapses to that in Kim et al. (2008). For more than

¹Kim et al. (2008) propose a version of their model for more than two regimes, but it is very restrictive in terms of the patterns of correlation between the state variable and the regression disturbance term that can be captured. In particular, their N -state model implies that larger positive values of the regression disturbance term are monotonically related to larger values of the state variable. Among other things, this makes results from this model highly dependent on the arbitrary decision of how the states are labeled.

two regimes, the model allows for a wide variety of patterns of correlation between the state variable and regression disturbance term. Despite this flexibility, the model maintains computational feasibility, and can be estimated via maximum likelihood using extensions to the filter in Hamilton (1989). The parameterization of the model also allows for a simple test of the null hypothesis of exogenous switching. Using simulation experiments, we demonstrate that the maximum likelihood estimator performs well in finite samples and that a likelihood ratio test of the null hypothesis of exogenous switching has good size and power properties.

We consider two applications of our N -regime endogenous switching model. In the first, we test for endogenous switching in a three regime switching mean model of U.S. real GDP growth. In the second, we consider endogenous switching inside of a three-regime version of the Turner et al. (1989) volatility feedback model of U.S. equity returns. We find statistically significant evidence of endogenous switching in both of these models, as well as quantitatively large differences in parameter estimates resulting from allowing for endogenous switching.

2 An N-State Endogenous Markov-Switching Model

Consider the following Gaussian regime-switching model:

$$y_t = g(x_t, y_{t-1}, \dots, y_{t-p}, S_t, S_{t-1}, \dots, S_{t-p}) + \sigma_{S_t} \varepsilon_t, \quad (1)$$

$$\varepsilon_t \sim i.i.d.N(0, 1),$$

where $g(\cdot)$ is a conditional mean function, y_t is scalar, x_t is a $k \times 1$ vector of observed exogenous variables, and $S_t \in \{0, 1, \dots, N - 1\}$ is an integer valued state variable indicating which of N different regimes is active at time t . Both y_t and x_t are assumed to be covariance stationary. Examples of equation (1) include a regime-switching regression model:

$$y_t = x_t' \beta_{S_t} + \sigma_{S_t} \varepsilon_t, \quad (2)$$

as well as a regime-switching autoregression:

$$y_t = \mu_{S_t} + \phi_1 (y_{t-1} - \mu_{S_{t-1}}) + \phi_2 (y_{t-2} - \mu_{S_{t-2}}) + \cdots + \phi_p (y_{t-p} - \mu_{S_{t-p}}) + \sigma_{S_t} \varepsilon_t. \quad (3)$$

For simplicity of exposition, we focus on the regime-switching regression model in equation (2) throughout this paper. However, the algorithms presented below for estimation and filtering are easily extended to the more general case of equation (1).

In an N -state Markov-switching model, the discrete regime indicator variable S_t follows an N -state Markov-process. Here we will allow the Markov-process to have time-varying transition probabilities as in Diebold et al. (1994) and Filardo (1994):

$$p_{ij,t} = \Pr(S_t = i | S_{t-1} = j, z_t) \quad (4)$$

In (4), the transition probability is influenced by the strictly exogenous or predetermined conditioning information in z_t , and is thus time varying. We assume that z_t is covariance stationary.

To model the dependence of the transition probability on z_t , it will be useful to alternatively describe S_t as the outcome of the values of $N - 1$ continuous latent variables, $S_{1,t}^*, S_{2,t}^*, \dots, S_{N-1,t}^*$. Here we use a multinomial probit specification for this purpose (see e.g. McCulloch et al. (2000)):

$$S_t = \left\{ \begin{array}{l} 0, \quad 0 = \max \{0, S_{1,t}^*, S_{2,t}^*, \dots, S_{N-1,t}^*\} \\ 1, \quad S_{1,t}^* = \max \{0, S_{1,t}^*, S_{2,t}^*, \dots, S_{N-1,t}^*\} \\ \vdots \\ N - 1, \quad S_{N-1,t}^* = \max \{0, S_{1,t}^*, S_{2,t}^*, \dots, S_{N-1,t}^*\} \end{array} \right\}, \quad (5)$$

where each of the $N - 1$ latent variables follow a symmetric process:

$$S_{i,t}^* = \gamma_{i,S_{t-1}} + z_t' \delta_{i,S_{t-1}} + \eta_{i,t} \quad (6)$$

$$\eta_{i,t} \sim i.i.d. N(0, 1)$$

$$i = 1, \dots, N - 1$$

This provides enough structure to parameterize the transition probabilities for the Markov-process:

$$p_{0j,t} = \Pr(\eta_{1,t} < -c_{1,j,t}, \eta_{2,t} < -c_{2,j,t}, \dots, \eta_{N-1,t} < -c_{N-1,j,t}), \quad (7)$$

$$p_{ij,t} = \Pr(-\eta_{i,t} < c_{i,j,t}, \{(\eta_{m,t} - \eta_{i,t}) < (c_{i,j,t} - c_{m,j,t}) : m = 1, \dots, N - 1, m \neq i\}), \quad (8)$$

where $c_{i,j,t} = \gamma_{i,j} + z_t' \delta_{i,j}$, $i = 1, \dots, N - 1$, $j = 1, \dots, N - 1$.

To model endogenous switching, we assume that the joint probability density for ε_t and $\eta_t = (\eta_{1,t}, \eta_{2,t}, \dots, \eta_{N-1,t})'$ is independent and identically distributed multivariate Gaussian:

$$\begin{bmatrix} \varepsilon_t \\ \eta_t \end{bmatrix} \sim i.i.d. N(\mathbf{0}_N, \Sigma), \quad (9)$$

where:

$$\Sigma = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \dots & \rho_{N-1} \\ \rho_1 & 1 & \rho_1 \rho_2 & \dots & \rho_1 \rho_{N-1} \\ \rho_2 & \rho_2 \rho_1 & 1 & \dots & \rho_2 \rho_{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{N-1} & \rho_{N-1} \rho_1 & \rho_{N-1} \rho_2 & \dots & 1 \end{bmatrix}. \quad (10)$$

This multivariate Gaussian density implies that the correlation between ε_t and $\eta_{i,t}$ is given by the parameter ρ_i . In turn, these ρ_i parameters control the extent of endogenous switching

in the model. Indeed, the exogenous switching model is nested through the parameter restriction $\rho_1 = \rho_2 = \dots = \rho_{N-1} = 0$. Further, the ρ_i parameters have a straightforward interpretation in terms of endogenous switching: When ρ_i is positive, larger values of ε_t are associated with an increased likelihood of $S_t = i$ occurring relative to $S_t = 0$. When $(\rho_i - \rho_m)$ is positive, larger values of ε_t are associated with an increased likelihood of $S_t = i$ occurring relative to $S_t = m$. The converse is also true. Note that nothing in this model takes a stand on the direction of causality, and the model could be consistent with causality running from ε_t to S_t , from S_t to ε_t , or bi-directional causality.

Note that the covariance matrix in (10) implies that conditional on ε_t , the $\eta_{i,t}$ are uncorrelated:

$$E(\eta_{\tau,t}\eta_{\tau',t}|\varepsilon_t) = 0, \quad \forall \tau \neq \tau'. \quad (11)$$

This assumption is required to identify the parameters of the model. Specifically, the covariance parameters in (11) are not separately identified from the $\gamma_{i,S_{t-j}}$ parameters in (6). Further, the assumption plays the role of a normalization, as these parameters are present only to parameterize the transition probabilities of the Markov process.

To gain further intuition into how our proposed endogenous Markov-switching model differs from the exogenous Markov-switching model, it is useful to consider the probability of transitioning between states, conditional on ε_t :

$$\tilde{p}_{ij,t} = \Pr(S_t = i | S_{t-1} = j, z_t, \varepsilon_t) \quad (12)$$

For the exogenous switching model, this conditional transition probability is equal to the unconditional transition probability, so that $\tilde{p}_{ij,t} = p_{ij,t}$. For the endogenous switching model the conditional and unconditional transition probabilities will not be equal, and the realization of ε_t can signal markedly different probabilities of transitioning between regimes.

As an example of this, Figure 1 plots unconditional and conditional transition probabilities against alternative realizations of $\varepsilon_t \in [-3, 3]$ for a particular parameterization of a three

state ($N = 3$) endogenous switching model. In this example, the dependence on z_t has been eliminated, and the correlation parameters have been set to $\rho_1 = -0.5$ and $\rho_2 = 0.9$. The figure shows that the conditional probability of transitioning regimes can vary in extreme directions depending on the outcome of ε_t . For example, focusing on the diagonal entries, the conditional probability of continuing in the $S_t = 0$ regime ($\tilde{p}_{00,t}$) increases gradually from around 0.3 to above 0.8 as ε_t moves from a large negative value (-3) toward 0. This transition probability then falls rapidly to near 0 as ε_t increases from 0 to around 2. The other continuation probabilities, $\tilde{p}_{11,t}$ and $\tilde{p}_{22,t}$, also display dramatic shifts that cover the entire probability range as ε_t is varied. Alternative parameterizations for ρ_1 and ρ_2 give alternative patterns of $\tilde{p}_{ij,t}$. An example of this is given in Figure 2, which depicts the transition probabilities when $\rho_1 = 0.9$ and $\rho_2 = 0.9$. These figures also demonstrate that the conditional transition probability can differ markedly from the unconditional transition probability, which is given by the horizontal dashed lines in each figure. As will be shown in detail in the next section, the ratio of these two probabilities is an important quantity in distinguishing the likelihood function for the endogenous switching model from that for the exogenous switching model.

3 Likelihood Calculation, State Filtering and Tests for Endogenous Switching

In this section we describe how both the likelihood function and filtered and smoothed probabilities of the states can be calculated for the endogenous switching model.² We will also describe how these calculations differ from those for the exogenous switching model. Finally, we discuss how tests of the null hypothesis of exogenous switching vs. the alternative hypothesis of endogenous switching can be conducted.

²The model likelihood for Markov-switching models will be invariant to an arbitrary relabeling of regimes. We assume throughout that the model has been appropriately normalized. Specific strategies for normalization will be discussed for the empirical analysis presented in Section 5.

Collect the model parameters into the vector θ , and let $\mathbf{Z}_t = \{z_t, z_{t-1}, \dots\}$ and $\Psi_t = \{y_t, y_{t-1}, \dots\}$ indicate the history of observed z_t and y_t through date t . As in Filardo (1994), the conditional likelihood value for y_t , $f(y_t|\Psi_{t-1}, \mathbf{Z}_t, \theta)$, $t = 1, \dots, T$, can be constructed recursively using an extension of the iterative formulas in Hamilton (1989) to the case of time-varying transition probabilities:³

$$f(y_t|\Psi_{t-1}, \mathbf{Z}_t, \theta) = \sum_{S_t=0}^{N-1} \sum_{S_{t-1}=0}^{N-1} f(y_t|S_t, S_{t-1}, \Psi_{t-1}, \mathbf{Z}_t, \theta) p_{ij,t} \Pr(S_{t-1}|\Psi_{t-1}, \mathbf{Z}_{t-1}, \theta) \quad (13)$$

$$\Pr(S_t = i|\Psi_t, \mathbf{Z}_t, \theta) \propto \sum_{S_{t-1}=0}^{N-1} f(y_t|S_t, S_{t-1}, \Psi_{t-1}, \mathbf{Z}_t, \theta) p_{ij,t} \Pr(S_{t-1}|\Psi_{t-1}, \mathbf{Z}_{t-1}, \theta) \quad (14)$$

These equations are then iterated recursively to obtain the log likelihood function $L(\theta) = \sum_{t=1}^T \log [f(y_t|\Psi_{t-1}, \mathbf{Z}_t, \theta)]$ and the filtered state estimates $\Pr(S_t = i|\Psi_t, \mathbf{Z}_t, \theta)$, $t = 1, \dots, T$. To initialize the recursion we require an initial filtered state probability, $\Pr(S_0 = i|\Psi_0, \mathbf{Z}_0, \theta)$, $i = 0, \dots, N - 1$, calculation of which can be quite involved. Here we follow the usual practice, suggested by Hamilton (1989), of approximating this initial probability with an unconditional probability. In the case of time-varying transition probabilities, we use the unconditional state probability computed assuming z_t is always at its sample mean. Denote this probability as $Pr(S_t = i|\bar{z})$, $i = 0, \dots, N - 1$, where \bar{z} is the sample mean of z_t . Next, define $\bar{p}_{ij} = \Pr(S_t = i|S_{t-1} = j, \bar{z})$, and collect these in a matrix of transition probabilities as:

$$\bar{P} = \begin{bmatrix} \bar{p}_{00} & \bar{p}_{01} & \dots & \bar{p}_{0N-1} \\ \bar{p}_{10} & \bar{p}_{11} & \dots & \bar{p}_{1N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{p}_{N-10} & \bar{p}_{N-11} & \dots & \bar{p}_{N-1N-1} \end{bmatrix} \quad (15)$$

³For notational convenience, we suppress the dependence of probability density functions on the regressors, x_t , throughout this section. Equations (13) and (14) make use of the assumption, implicit in equation (2), that conditional on x_t and the state indicator S_t , the probability density function of y_t does not depend on z_t . This is without loss of generality, since x_t may include elements of z_t .

Finally, define:

$$A = \begin{bmatrix} I_N - \bar{P} \\ \iota'_N \end{bmatrix}$$

where I_N is the $N \times N$ identity matrix and ι_N is an $N \times 1$ vector of ones. The vector holding $Pr(S_t = i | \bar{z})$, $i = 0, \dots, N-1$ is then computed as the last column of the matrix $(A'A)^{-1} A'$.

The key element required to compute each step of the the recursion in (13) and (14) is $f(y_t | S_t, S_{t-1}, \Psi_{t-1}, \mathbf{Z}_t, \theta)$, and it is here that we see the distinction in the likelihood function between the exogenous and endogenous switching models. In the exogenous switching model, the state indicators $S_t = i$ and $S_{t-1} = j$ simply define the mean and variance of a Gaussian distribution for y_t , such that:

$$f(y_t | S_t = i, S_{t-1} = j, \Psi_{t-1}, \mathbf{Z}_t, \theta) = \frac{1}{\sigma_i} \phi\left(\frac{y_t - x'_t \beta_i}{\sigma_i}\right)$$

where $\phi()$ indicates the standard normal probability density function. By contrast, when there is endogenous switching, the state variables $S_t = i$ and $S_{t-1} = j$ indicate not just the parameters of the relevant data generating process, but additionally provide information about which values of the random disturbance, ε_t , are most likely. In the case of endogenous switching:

$$f(y_t | S_t = i, S_{t-1} = j, \Psi_{t-1}, \mathbf{Z}_t, \theta) = \frac{\tilde{p}_{ij,t}}{p_{ij,t}} \left[\frac{1}{\sigma_i} \phi\left(\frac{y_t - x'_t \beta_i}{\sigma_i}\right) \right] \quad (16)$$

This equation, which is derived in Appendix A.1, can be interpreted as follows. The term in brackets is the regime-dependent conditional density of y_t for the exogenous switching model. This density is then weighted by a ratio of probabilities of transitioning from regime j to regime i , where the probability in the numerator is conditional on the regime-specific value of ε_t and the probability in the denominator is not. The unconditional transition prob-

ability $p_{ij,t}$ can be interpreted as the average value of $\tilde{p}_{ij,t}$ with respect to the unconditional distribution of ε_t . In other words, $p_{ij,t}$ gives the average probability of transitioning from state j to state i with respect to ε_t . Thus, equation (16) says that if the value of ε_t signals an above average probability of transitioning from state j to state i , then the likelihood value for y_t conditional on $S_t = i$ and $S_{t-1} = j$ will be higher than would be calculated under the exogenous switching model. Returning to Figures 1 and 2, the ratio $\tilde{p}_{ij,t}/p_{ij,t}$ can be far from unity, meaning the likelihood function for the exogenous switching model may be substantially misspecified in the presence of endogenous switching. In general, estimation assuming exogenous switching will lead to biased parameter estimates as well as biased filtered and smoothed state probabilities when endogenous switching is present.

The recursion provided by equations (13) and (14) can be used to construct the value of the likelihood function for any value of θ . The likelihood function can then be numerically maximized with respect to θ to obtain the maximum likelihood estimates, $\hat{\theta}$.⁴ Given these estimates, the recursion can be run again to provide the filtered state probability evaluated at the maximum likelihood estimates, $\Pr(S_t = i | \Psi_t, \mathbf{Z}_t, \hat{\theta})$. In many applications we also require the so-called “smoothed” state probability $\Pr(S_t = i | \Psi_T, \mathbf{Z}_T, \theta)$, which provides inference on S_t conditional on all available sample information. To compute the smoothed probabilities, we can apply the recursive filter provided in Kim and Nelson (1999b), which remains valid for the N-state endogenous Markov-switching model described in Section 2. Beginning with the final filtered probability, $\Pr(S_T = j | \Psi_T, \mathbf{Z}_T, \theta)$, $j = 0, \dots, N - 1$, the following equation can be applied recursively, for $t = T - 1, \dots, 1$:

$$\Pr(S_t = i | \Psi_T, \mathbf{Z}_T, \theta) = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \Pr(S_{t-1} = j, S_t = i, S_{t+1} = k | \Psi_T, \mathbf{Z}_T, \theta) \quad (17)$$

⁴One practical computational difficulty in constructing the likelihood function is that it requires computing the unconditional and conditional transition probabilities, $\tilde{p}_{ij,t}$ and $p_{ij,t}$, which involves calculation of a multivariate Gaussian cumulative distribution function (CDF), for which there is no closed form solution. In our empirical implementation of the endogenous switching model we use Matlab’s “mvncdf” command to numerically compute the required integrals. Appendix A.2 provides an explicit characterization of these CDFs for the case of $N=3$.

where:

$$\begin{aligned} & \Pr(S_{t-1} = j, S_t = i, S_{t+1} = k | \Psi_T, \mathbf{Z}_T, \theta) \\ &= \frac{\Pr(S_t = i, S_{t+1} = k | \Psi_T, \mathbf{Z}_T, \theta) p_{ki,t} \Pr(S_t = i, S_{t-1} = j | \Psi_t, \mathbf{Z}_t, \theta)}{\Pr(S_t = j, S_{t+1} = k | \Psi_t, \mathbf{Z}_t, \theta)} \end{aligned} \quad (18)$$

For additional details of the derivation of equation (18), see Kim (1994) and Kim and Nelson (1999b).

To conclude this section, we describe how statistical hypothesis tests of the null hypothesis of exogenous switching can be conducted. Our N -state endogenous switching model collapses to a standard exogenous Markov-switching model in the case where:

$$\rho_1 = \rho_2 = \dots = \rho_{N-1} = 0, \quad (19)$$

Thus, the null hypothesis of exogenous switching can be tested by any suitable joint test of the $N - 1$ zero restrictions in 19. In the simulation studies presented in Section 4, we will consider the finite sample performance of both Wald and likelihood ratio tests of these restrictions.

4 Monte Carlo Evidence

In this section we describe results from a Monte Carlo simulation study. We first evaluate the finite sample performance of the maximum likelihood estimator (MLE) applied to data generated from an endogenous switching model. We also evaluate the size and power performance of hypothesis tests for endogenous switching. Finally, we investigate the ability of an information criteria approach to determine the number of regimes in the presence of endogenous switching.

To focus on the results most germane to the addition of endogenous switching, we consider a simplified version of the general model presented in Section 2. In particular, we focus on

the Gaussian Markov-switching mean model:

$$y_t = \mu_{S_t} + \sigma \varepsilon_t \tag{20}$$

$$\varepsilon_t \sim i.i.d.N(0, 1)$$

where $S_t \in \{0, 1, 2\}$ is a three-state Markov process that evolves with fixed transition probabilities $p_{ij} = \Pr(S_t = i | S_{t-1} = j)$. In all Monte Carlo simulations we set $\mu_{S_t} \in \{-1, 0, 1\}$ and $\sigma = 1$. The Markov process evolves according to the endogenous switching model outlined in Section 2 with $z_t = 0, \forall t$. Across alternative Monte Carlo experiments we vary the persistence of the transition probabilities for remaining in a regime from a “high persistence” case ($p_{00} = p_{11} = p_{22} = 0.9$) to a “low persistence” case ($p_{00} = p_{11} = p_{22} = 0.7$). In both the high and low persistence cases we spread the residual probability evenly across the remaining transitions. We vary the size of the correlation parameters from $\rho_1 = \rho_2 = 0.9$ to $\rho_1 = \rho_2 = 0.5$. Finally, we consider two sample sizes, $T = 300$ and $T = 500$. Performance is measured using the mean and root mean squared error (RMSE) of the estimates of each parameter across 1000 Monte Carlo simulations. The RMSE, reported in parentheses, is computed relative to the true value for each parameter.

Table 1 presents results regarding the performance of the MLE that incorrectly assumes exogenous switching, and demonstrates that the bias in this incorrectly specified MLE can be severe. The bias in the μ_i parameters increases as the state persistence falls, with the amount of bias reaching as high as 67% of the true parameter value in the case of μ_0 . Estimation bias is also visible in the estimates of the conditional variance term, with the bias in some cases above 15% of the true parameter value. The estimation bias is not a small sample phenomenon, with similar bias observed for $T = 300$ as for $T = 500$. The bias decreases as the correlation parameters, ρ_1 and ρ_2 , fall from 0.9 to 0.5. However, despite this substantially lessened importance of endogenous switching, the MLE that ignores endogenous switching still generates very biased parameter estimates, with bias reaching as high as 43% of the

true parameter value for μ_0 .

Table 2 shows results for the same variety of data generating processes, but with the MLE now applied to the correctly specified model. These results demonstrate that the MLE of the correctly specified model performs very well, with mean parameter estimates that are close to the true value, and RMSE statistics that are small. The performance of the correctly specified estimator is largely unaffected by the extent of state persistence or the value of the correlation parameters. The sample size also does not have large effects on the mean estimates although, not surprisingly, the RMSE is higher when the sample size is smaller.

Table 3 shows results of simulations meant to assess the finite sample performance of both Wald and likelihood ratio (LR) tests of the null hypothesis of exogenous switching, which is parameterized as a test of the joint restriction $\rho_1 = \rho_2 = 0$. We again consider two sample sizes, as well as a high and low state persistence case. To evaluate the size of the Wald and LR tests, we first consider the case where the true data generating process has $\rho_1 = \rho_2 = 0$. To evaluate the power of these tests we consider two cases, one in which the extent of endogenous switching is high ($\rho_1 = \rho_2 = 0.9$) and the other where endogenous switching is more moderate ($\rho_1 = \rho_2 = 0.5$). The size results are based on rejection rates of 5%-level tests using asymptotic critical values. The power results are based on rejection rates using size-adjusted 5% critical values.

Beginning with the size of the tests, both the Wald and LR test are moderately oversized when $T=300$, with rejection rates around 10% for the Wald test and 8% for the LR test. Not surprisingly, the empirical size is closer to the level implied by the asymptotic critical values when the sample size is larger. The LR test in particular has empirical size very close to 5% when $T=500$. Turning to the power results, the LR test displays consistently high rejection rates ranging between 69% and 100%. The Wald test is less powerful, with rejection rates ranging from 42% to 90%.

Our discussion to this point has focused on estimation of models where the number of Markov-switching states, N , is assumed known. In practice however, the value of N is a

matter of model selection, and the selection of N in the presence of endogenous switching has not been previously discussed in the literature. Here we propose the use of the Markov-switching information criteria (MSC) developed by Smith et al. (2006) for this purpose. Although the MSC was developed under the assumption of an exogenous switching model, it is not unreasonable to expect that it would have power to detect the number of states under endogenous switching. To assess this, we evaluate the performance of the MSC with simulated datasets.

The MSC is defined as follows:

$$MSC = -2L(\hat{\theta}) + \sum_{i=1}^N \frac{\hat{T}_i (\hat{T}_i + \lambda_i k)}{\delta_i \hat{T}_i - \lambda_i k - 2}, \quad (21)$$

where $L(\hat{\theta})$ is the maximized log-likelihood value for the endogenous switching model with N states, and $\hat{T}_i = \sum_{t=1}^T \Pr(S_t = i | \Psi_T, \mathbf{Z}_T, \hat{\theta})$ denotes the summation over time of the smoothed probabilities of each state, $i = 1, \dots, N$. For λ_i and δ_i , we follow the suggestion of Smith et al. (2006) and set these to N and 1 respectively. Finally, k denotes the number of explanatory variables. The MSC is calculated for alternative values of N , and the selected value of N is that which produces the minimum value of the MSC.

Table 4 shows results of application of the MSC to select N in simulated datasets, where for each dataset the MSC is calculated for values of N from 2 to 4. The true data generating process is the switching intercept model with $N = 3$ in (20). We again consider two sample sizes, a high and low state persistence case, and a high and low case for the size of the endogenous switching correlation parameters. The entries in Table 4 show the percentage of simulated datasets for which the MSC was minimized at the indicated value of N .

The results in Table 4 demonstrate that the MSC performs very well for selecting the number of states in an endogenous switching data generating process. Specifically, the MSC selects the correct value of N over 97% of the time for all versions of the data generating process considered. It is notable that the performance of the MSC does not deteriorate as the

correlation parameters (ρ_1 and ρ_2) increase, suggesting that endogenous switching does not adversely affect the MSC over the exogenous switching process the criterion was originally designed for. The extent of state persistence does affect the performance of the MSC, with the MSC selecting the correct number of states more often as the state process becomes more persistent. This is expected as, all else equal, the states become easier to detect as their persistence increases.

In summary, the Monte Carlo results suggest that ignoring endogenous switching can lead to substantial bias in the MLE when endogenous switching is in fact present. This bias persists into large sample sizes, and for both high and moderate values of the parameters that control the extent of endogenous switching. The MLE that accounts for endogenous switching performed very well, yielding accurate parameter estimates. Also, the LR test for exogenous switching was effective, with approximately correct size and good power. Finally, the MSC performed very well as an approach to select the number of Markov-switching states in an endogenous switching process.

5 Applications in Macroeconomics and Finance

In this section, we consider two applications of the N-state endogenous Markov-switching model. In Section 5.1, we consider endogenous switching in a three-state model of U.S. business cycle dynamics. In Section 5.2 we extend the two-state endogenous-switching volatility feedback model of U.S. equity returns in Kim et al. (2008) to allow for three volatility regimes.

5.1 U.S. Business Cycle Fluctuations

One empirical characteristic of the U.S. business cycle highlighted by Burns and Mitchell (1946) is asymmetry in the behavior of real output across business cycle phases. In his seminal paper, Hamilton (1989) captures asymmetry in the business cycle using a two-state Markov-switching autoregressive model of U.S. real GNP growth. His model identifies one

phase as relatively brief periods of steep declines in output, and the other as relatively long periods of gradual output increases. Using quarterly data from 1952:Q2 to 1984:Q4, Hamilton (1989) shows that the estimated shifts between the two phases accord well with the National Bureau of Economic Research (NBER) chronology of U.S. business cycle peaks and troughs.

While Hamilton’s original model captures the short and steep nature of recessions relative to expansions, it does not incorporate an important feature of the business cycle that was prevalent over the sample period he considered: recessions were typically followed by high-growth recovery phases that pushed output back toward its pre-recession level. This “bounce back” effect is evident in the post-recession real GDP growth rates shown in Table 5. In order to capture this high growth recovery phase, Sichel (1994) and Boldin (1996) extend Hamilton’s original model to a three state Markov-switching model.

Here we also use a three state Markov-switching model to capture mature expansions, recessions, and a post-recession expansion phase. In particular, we assume that the U.S. real GDP growth rate is described by the following three state Markov-switching mean model:

$$\begin{aligned}\Delta y_t &= \mu_{S_t} + \sigma \varepsilon_t \\ \varepsilon_t &\sim i.i.d.N(0, 1)\end{aligned}\tag{22}$$

where y_t is U.S. log real GDP and $S_t \in \{0, 1, 2\}$ is a Markov-switching state variable that evolves with fixed transition probabilities p_{ij} . Note that in this model U.S. real GDP growth follows a white noise process inside of each regime. This intra-regime lack of dynamics is consistent with the results of Kim et al. (2005) and Camacho and Perez-Quiros (2007), who find that traditional linear autoregressive dynamics in U.S. real GDP growth are largely absent once mean growth is allowed to follow a three-regime Markov-switching process.⁵

⁵Although the number of Markov-switching states follows from the definition of the business cycle phases described above, we have also used the MSC to select the number of states in a data-based manner. The MSC selects the three-state model over a two-state model of the business cycle for U.S real GDP growth.

We restrict the model in two ways. First, we restrict $\mu_0 > 0$, $\mu_1 < 0$, and $\mu_2 > 0$, which serves to identify $S_t = 1$ as the recession regime, and $S_t = 0$ and $S_t = 2$ as expansion regimes. Second, following Boldin (1996), we restrict the matrix of transition probabilities so that the states occur in the order $0 \rightarrow 1 \rightarrow 2$:

$$\bar{P} = \begin{bmatrix} p_{00} & 0 & 1 - p_{22} \\ 1 - p_{00} & p_{11} & 0 \\ 0 & 1 - p_{11} & p_{22} \end{bmatrix},$$

In combination with the restrictions on μ_{S_t} , this form of the transition matrix restricts the regimes to occur in the order: mature expansion \rightarrow recession \rightarrow post-recession expansion. We will consider two versions of this model, one in which the Markov-switching is assumed to be exogenous, so that $\rho_1 = \rho_2 = 0$, and one that allows for endogenous switching.

We estimate this model using data on quarterly U.S. real GDP growth from 1947:Q1 to 2016:Q2. Over this sample period, there are two prominent types of structural change in the U.S. business cycle that are empirically relevant. The first is the well-documented reduction in real GDP growth volatility in the early 1980s known as the ‘‘Great Moderation’’ (Kim and Nelson (1999a), McConnell and Perez-Quiros (2000)). To capture this reduction in volatility, we include a one time change in the conditional volatility parameter, σ , in 1984:Q1, the date identified by Kim and Nelson (1999a) as the beginning of the Great Moderation. The second, as identified in Kim and Murray (2002) and Kim et al. (2005), is the lack of a high growth recovery phase following the three most recent NBER recessions. To capture this change in post-recession growth rates, we include a one-time break in μ_2 . Finally, to allow for the possibility that the nature of endogenous switching changed along with the nature of the post-recession regime, we allow for breaks in ρ_1 and ρ_2 . These breaks in μ_2 , ρ_1 and ρ_2 are also dated to 1984:Q1, although results are insensitive to alternative break dates between 1984:Q1 and the beginning of the 1990-1991 recession. All other model parameters are assumed to be constant over the entire sample period.

The second and third columns of Table 6 shows the maximum likelihood estimation results when we assume exogenous switching. The estimates show a prominent high growth recovery phase before 1984:Q1 ($\mu_{2,1} \gg \mu_0$). The estimates also show that this high growth recovery phase has disappeared in recent recessions, and indeed been replaced with a *low-growth* post-recession phase ($\mu_{2,2} < \mu_0$). The conditional volatility parameter, σ , falls by 50% after 1984, consistent with the large literature on the Great Moderation.

The maximum likelihood estimates assuming endogenous switching are shown in the fourth and fifth columns of Table 6. A likelihood ratio test rejects the null hypothesis of exogenous switching at the 5% level (p-value = 0.045). The estimates of the correlation parameters prior to 1984 are such that $\rho_2 < \rho_1 < 0$. This pattern of correlations means that larger positive values of ε_t increase the likelihood of $S_t = 0$ (mature expansion) relative to $S_t = 1$ (recession) and $S_t = 2$ (post-recession expansion), and increase the likelihood of $S_t = 1$ relative to $S_t = 2$. These estimates switch signs after 1984, such that $\rho_1 \approx \rho_2 > 0$. In this case, larger positive values of ε_t increase the likelihood of $S_t = 1$ and $S_t = 2$ relative to $S_t = 0$.

There is also evidence of bias in the parameter estimates of the exogenous switching model. The estimated mean growth rate in the post-recession expansion phase is substantially different when accounting for endogenous switching. Also, the continuation probability for the post-recession phase, p_{22} , is substantially lower when accounting for endogenous switching, meaning the length of these phases are overstated by the results from exogenous switching models. Finally, results of the Ljung-Box test, shown in the bottom panel of the table, show that accounting for endogenous switching eliminates autocorrelation in the disturbance term that is present in the exogenous switching model.

Figure 3 displays the smoothed state probabilities for both the exogenous and endogenous switching models, and shows the distortion in estimated state probabilities that can occur from ignoring endogenous switching. From panel (a), we see that the smoothed probability that the economy is in a mature expansion ($S_t = 0$) is often lower for the exogenous switching

model than the endogenous switching model, while panel (c) shows that the opposite is true for the smoothed probability of the post-recession expansion phase ($S_t = 2$). Put differently, the endogenous switching model suggests a quicker transition from the post-recession expansion phase to the mature expansion phase than does the exogenous switching model.

5.2 Volatility Regimes in U.S. Equity Returns

An empirical regularity of U.S. equity returns is that low returns are contemporaneously associated with high volatility. This is a counterintuitive result, as classical portfolio theory implies the equity risk premium should respond positively to the expectation of future volatility. One explanation for this observation is that while investors do require an increase in expected return for expected future volatility, they are often surprised by news about realized volatility. This “volatility feedback” creates a reduction in prices in the period in which the increase in volatility is realized. The volatility feedback effect has been investigated extensively in the literature by French et al. (1987), Turner et al. (1989), Campbell and Hentschel (1992), Bekaert and Wu (2000) and Kim et al. (2004).

Turner et al. (1989) (TSN hereafter) model the volatility feedback effect with a two state Markov-switching model:

$$r_t = \theta_1 E(\sigma_{S_t}^2 | I_{t-1}) + \theta_2 [E(\sigma_{S_t}^2 | I_t^*) - E(\sigma_{S_t}^2 | I_{t-1})] + \sigma_{S_t} u_t$$

$$u_t \sim i.i.d.N(0, 1)$$

where r_t is a measure of excess equity returns, $I_t = \{r_t, r_{t-1}, \dots\}$, and I_t^* is an information set that includes I_{t-1} and the information investors observe during period t . $S_t \in \{0, 1\}$ is a discrete variable that follows a two state Markov process with fixed transition probabilities p_{ij} . To normalize the model, TSN restrict $\sigma_1 > \sigma_0$, so that state 1 is the higher volatility regime.

One estimation difficulty with the above model is that there exists a discrepancy between the investors' and the econometrician's information set. In particular, while I_{t-1} may be summarized by returns up to period $t - 1$, the information set I_t^* includes information that is not summarized in the econometrician's data set on observed returns. To handle this estimation difficulty, TSN use actual volatility, $\sigma_{S_t}^2$ to approximate $E(\sigma_{S_t}^2 | I_t^*)$. That is, they estimate:

$$r_t = \theta_1 E(\sigma_{S_t}^2 | I_{t-1}) + \theta_2 [\sigma_{S_t}^2 - E(\sigma_{S_t}^2 | I_{t-1})] + \sigma_{S_t} \varepsilon_t \quad (23)$$

$$\varepsilon_t \sim i.i.d.N(0, 1)$$

Kim, Piger, and Startz (2008) (KPS, hereafter) point out that this approximation introduces classical measurement error into the state variable in the estimated equation, thus rendering it endogenous. KPS propose a two-state endogenous Markov-switching model to deal with this endogeneity problem. Again, this two-state model of endogenous switching is identical to the N -state endogenous switching model proposed in Section 2 when $N = 2$. However, there is substantial evidence for more than two volatility regimes in U.S. equity returns (Guidolin and Timmermann (2005)). Here, we extend the TSN and KPS exogenous and endogenous switching volatility feedback models to allow for three volatility regimes. Specifically, we extend the volatility feedback model in equation (23) to allow for three regimes, $S_t \in \{0, 1, 2\}$, with fixed transition probabilities p_{ij} . For normalization we assume $\sigma_2 > \sigma_1 > \sigma_0$, so that state 2 is the highest volatility regime. Note that the extension to three states is supported by the data. Specifically, the MSC selects $N = 3$ as the preferred number of states for both the exogenous and endogenous switching volatility feedback model.

To estimate the three state volatility feedback model, we measure excess equity returns using monthly returns for a value-weighted portfolio of all NYSE-listed stocks in excess of the one-month Treasury Bill rate. The sample period extends from January 1952 to December 2015. The second and third columns of Table 7 show the estimation results when we assume

exogenous switching. The estimates are consistent with a positive relationship between the risk premium and expected future volatility ($\theta_1 > 0$) and a substantial volatility feedback effect ($\theta_2 \ll 0$). The estimates also suggest a dominant volatility feedback effect, as θ_1 is small in absolute value relative to θ_2 . The Ljung-Box test, shown in the middle panel of the table, shows an absence of autocorrelation in the disturbance terms. However, results of the ARCH-LM test, shown in the bottom panel, shows some correlation remaining in the squared residuals, suggesting that the three-state volatility feedback model with exogenous switching is not sufficient to entirely capture the persistence in conditional volatility.

The fourth and fifth columns of Table 7 show the results when we allow for endogenous switching. First, there is statistically significant evidence in favor of endogenous switching, with a likelihood ratio test rejecting the null hypothesis of exogenous switching at the 5% level (p-value = 0.034). The primary difference in the estimated parameters is a much smaller volatility feedback effect (smaller θ_2) in the endogenous switching model than was found in the exogenous switching model. The estimated correlation parameters have different signs, with $\rho_1 < 0$ and $\rho_2 > 0$. Thus, large values of ε_t in equation (23) increase the likelihood of $S_t = 0$ (low volatility regime) relative to $S_t = 1$ (medium volatility regime), and increase the likelihood of $S_t = 2$ (highest volatility regime) relative to both $S_t = 0$ and $S_t = 1$. Unlike the exogenous switching version, the ARCH-LM test suggests an absence of correlation in the squared residuals of the three-state endogenous Markov-switching volatility feedback model. Thus, this model appears sufficient to capture the conditional volatility of excess returns.⁶

Figure 4 shows the risk premium implied by three different volatility feedback models, the exogenous switching model with three states (red dashed line), the endogenous switching model with three states (blue solid line), and the endogenous switching model with two states (green dotted line). The three state endogenous switching model produces a risk premium that is more variable than the other models across volatility states. In particular, the risk

⁶We have also estimated two-state volatility feedback models (not reported). The ARCH-LM test for both the endogenous and exogenous versions of these models suggests there is remaining serial correlation in the squared residuals. This is additional evidence for the three-state version of the Markov-switching volatility feedback model vs. the two state version.

premium from the three state endogenous switching model rises above the risk premium from the other models during the highest volatility state, which from Figure 5 is seen to be highly correlated with NBER recessions. However, during the other volatility states, the risk premium from the three state endogenous switching model is generally below that from the other models. On average, our model suggests a 9% risk premium, similar to that estimated by Kim et al. (2004) using the volatility feedback model assuming exogenous switching over the period 1952 to 1999. This estimated risk premium is higher than Fama and French (2002), who estimate an average risk premium of 2.5% using the average dividend yield plus the average dividend growth rate for the S&P 500 index over the period 1951 to 2000.

Finally, it is interesting to assess whether our introduction of additional, endogenous switching, states produces improved out-of-sample forecasts. To investigate this, we estimate four versions of the Markov-switching volatility feedback model. The models differ in the assumption regarding the number of Markov-switching states (two vs. three) and the assumption regarding whether these states are exogenous vs. endogenous. We then produce one-month-ahead forecasts of excess equity returns from these models over an out-of-sample period that extends from January 2000 through the end of our sample period, December 2015. Forecasts are produced based on models estimated over expanding windows.⁷

Table 8 gives the results of this out-of-sample forecasting experiment, reported as the root mean squared error (RMSE) for each model over the out-of-sample period. Overall, these results are consistent with the in-sample evidence favoring the three-state specification with endogenous switching. Specifically, the results document RMSE reductions achieved by moving from a two to three state version of the model, regardless of whether the switching is exogenous or endogenous. Also, the RMSE falls when moving from exogenous to endogenous switching models, regardless of the number of states. Consistent with the in-sample evidence, the best forecasting performance comes from the three-state model with endogenous switching.

⁷Appendix A.3 details how one-step-ahead forecasts can be constructed for both the exogenous and endogenous Markov-switching volatility feedback model.

6 Conclusion

We proposed a novel N -state Markov-switching regression model in which the state indicator variable is correlated with the regression disturbance term. The model admits a wide variety of patterns for this correlation, while maintaining computational feasibility. Maximum likelihood estimation can be performed using extensions to the filter in Hamilton (1989), and the parameterization of the model allows for a simple test of the null hypothesis of exogenous switching. In simulation experiments, the maximum likelihood estimator performed well, and a likelihood ratio test of exogenous switching had good size and power properties. We allowed for endogenous switching in two applications: a switching mean model of U.S. real GDP, and a switching volatility model of U.S. equity returns. We find statistically significant evidence of endogenous switching in both of these models, as well as quantitatively large differences in parameter estimates resulting from allowing for endogenous switching.

Appendix A.1: Derivation of $f(y_t|S_t, S_{t-1}, \Psi_{t-1}, \mathbf{Z}_t, \theta)$

The iterative filter presented in Section 3 requires calculation of the regime-dependent density $f(y_t|S_t, S_{t-1}, \Psi_{t-1}, \mathbf{Z}_t, \theta)$, where y_t represents the random variable described by the data generating process described in equation (2) along with the endogenous regime-switching process described in Section 2. We have again suppressed the conditioning of this density on the covariates x_t . This appendix derives this regime-dependent density.

Let y_t^* denote a realization of y_t for which we wish to compute $f(y_t^*|S_t = i, S_{t-1} = j, \Psi_{t-1}, \mathbf{Z}_t, \theta)$. Applying Bayes Rule yields:

$$f(y_t|S_t = i, S_{t-1} = j, \Psi_{t-1}, \mathbf{Z}_t, \theta) = \frac{f(y_t, S_t = i|S_{t-1} = j, \Psi_{t-1}, \mathbf{Z}_t, \theta)}{\Pr(S_t = i|S_{t-1} = j, \Psi_{t-1}, \mathbf{Z}_t, \theta)} \quad (\text{A.1})$$

The denominator of equation (A.1) is the time-varying transition probability, $p_{ij,t}$. Consider the following CDF of the numerator of (A.1):

$$\begin{aligned} & \Pr(y_t < y_t^*, S_t = i|S_{t-1} = j, \Psi_{t-1}, \mathbf{Z}_t) \\ &= \int_{-\infty}^{y_t^*} f(y_t, S_t = i|S_{t-1} = j, \Psi_{t-1}, \mathbf{Z}_t) dy_t \\ &= \int_{-\infty}^{\frac{y_t^* - x_t' \beta_i}{\sigma_i}} f(\varepsilon_t, S_t = i|S_{t-1} = j, \Psi_{t-1}, \mathbf{Z}_t) d\varepsilon_t \\ &= \int_{-\infty}^{\frac{y_t^* - x_t' \beta_i}{\sigma_i}} \Pr(S_t = i|\varepsilon_t, S_{t-1} = j, \Psi_{t-1}, \mathbf{Z}_t) f(\varepsilon_t|S_{t-1} = j, \Psi_{t-1}, \mathbf{Z}_t) d\varepsilon_t \\ &= \int_{-\infty}^{\frac{y_t^* - x_t' \beta_i}{\sigma_i}} \Pr(S_t = i|\varepsilon_t, S_{t-1} = j, \Psi_{t-1}, \mathbf{Z}_t) f(\varepsilon_t) d\varepsilon_t \end{aligned}$$

where the validity of moving to the last line in this derivation is ensured by the independence of ε_t over time, the exogeneity of \mathbf{Z}_t , and the independence of ε_t and S_{t-1} . Finally,

differentiating this CDF with respect to y_t^* yields:

$$\begin{aligned} & f(y_t^*, S_t = i | S_{t-1} = j, \Psi_{t-1}, \mathbf{Z}_t, \theta) \\ &= \Pr\left(S_t = i \mid \left(\frac{y_t^* - x_t' \beta_i}{\sigma_i}\right), S_{t-1} = j, \Psi_{t-1}, \mathbf{Z}_t\right) f\left(\frac{y_t^* - x_t' \beta_i}{\sigma_i}\right) \end{aligned} \quad (\text{A.2})$$

where $(y_t^* - x_t' \beta_i) / \sigma_i$ is a realization of the random variable ε_t . The first term in (A.2) is the conditional transition probability, $\tilde{p}_{ij,t}$. Given the marginal Gaussian distribution for ε_t , the second term in equation (A.2) is:

$$f\left(\frac{y_t^* - x_t' \beta_i}{\sigma_i}\right) = \frac{1}{\sigma_i} \phi\left(\frac{y_t^* - x_t' \beta_i}{\sigma_i}\right)$$

Combining the above results, we have:

$$f(y_t^* | S_t = i, S_{t-1} = j, \Psi_{t-1}, \mathbf{Z}_t, x_t) = \frac{\tilde{p}_{ij,t}}{p_{ij,t}} \left[\frac{1}{\sigma_i} \phi\left(\frac{y_t^* - x_t' \beta_i}{\sigma_i}\right) \right]$$

which is equation (16) evaluated at y_t^* .

Appendix A.2: Transition Probabilities for a Three-State Endogenous Markov-Switching Model

The unconditional and conditional transition probabilities defined in Section 2 take the form of a multivariate normal cumulative distribution function (CDF). In this appendix we explicitly characterize these CDFs for the case where $N=3$. First, define $c_{1,j,t} = \gamma_{1,j} + z'_t \delta_{1,j}$ and $c_{2,j,t} = \gamma_{2,j} + z'_t \delta_{2,j}$. For unconditional transition probabilities, we have the following results:

1. $\Pr(S_t = 0 | S_{t-1} = j, z_t) = p_{0j,t} = \Pr(\eta_{1,t} < -c_{1,j,t}, \eta_{2,t} < -c_{2,j,t})$ is calculated using the CDF of the following multivariate normal distribution:

$$\begin{pmatrix} \eta_{1,t} \\ \eta_{2,t} \end{pmatrix} \sim N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho_1 \rho_2 \\ \rho_1 \rho_2 & 1 \end{bmatrix} \right)$$

2. $\Pr(S_t = 1 | S_{t-1} = j, z_t) = p_{1j,t} = \Pr(-\eta_{1,t} < c_{1,j,t}, (\eta_{2,t} - \eta_{1,t}) < (c_{1,j,t} - c_{2,j,t}))$ is calculated using the CDF of the following multivariate normal distribution:

$$\begin{pmatrix} -\eta_{1,t} \\ \eta_{2,t} - \eta_{1,t} \end{pmatrix} \sim N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 - \rho_1 \rho_2 \\ 1 - \rho_1 \rho_2 & 2(1 - \rho_1 \rho_2) \end{bmatrix} \right)$$

3. Finally, $\Pr(S_t = 2 | S_{t-1} = j, z_t) = p_{2j,t}$ is calculated as:

$$p_{2j,t} = 1 - p_{0j,t} - p_{1j,t}$$

For conditional transition probabilities, we have the following results:

1. $\Pr(S_t = 0 | S_{t-1} = j, z_t, \varepsilon_t) = \tilde{p}_{0j,t} = \Pr(\eta_{1,t} < -c_{1,j,t}, \eta_{2,t} < -c_{2,j,t} | \varepsilon_t)$ can be calculated as:

$$\begin{aligned} \tilde{p}_{0j,t} &= \Pr(\eta_{1,t} < -c_{1,j,t} | \varepsilon_t) \Pr(\eta_{2,t} < -c_{2,j,t} | \varepsilon_t) \\ &= \Phi\left(\frac{-c_{1,j,t} - \rho_1 \varepsilon_t}{\sqrt{1 - \rho_1^2}}\right) \Phi\left(\frac{-c_{2,j,t} - \rho_2 \varepsilon_t}{\sqrt{1 - \rho_2^2}}\right) \end{aligned}$$

where $\Phi(\cdot)$ denotes the CDF of the univariate standard normal density. This is justified by the conditional independence assumption detailed in equation (11).

2. $\Pr(S_t = 1 | S_{t-1} = j, z_t, \varepsilon_t) = \tilde{p}_{1j,t} = \Pr(-\eta_{1,t} < c_{1,j,t}, (\eta_{2,t} - \eta_{1,t}) < (c_{1,j,t} - c_{2,j,t}) | \varepsilon_t)$ is calculated as the CDF of the following multivariate normal density:

$$\begin{pmatrix} -\eta_{1,t} \\ \eta_{2,t} - \eta_{1,t} \end{pmatrix} \Bigg| \varepsilon_t \sim N \left(\begin{bmatrix} -\rho_1 \varepsilon_t \\ (\rho_2 - \rho_1) \varepsilon_t \end{bmatrix}, \begin{bmatrix} 1 - \rho_1^2 & 1 - \rho_1^2 \\ 1 - \rho_1^2 & 2 - \rho_1^2 - \rho_2^2 \end{bmatrix} \right)$$

3. Finally, $\Pr(S_t = 2 | S_{t-1} = j, z_t, \varepsilon_t) = \tilde{p}_{2j,t} = \Pr((\eta_{1,t} - \eta_{2,t}) < (c_{2,j,t} - c_{1,j,t}), -\eta_{2,t} < c_{2,j,t} | \varepsilon_t)$ is calculated as the CDF of the following multivariate normal density:

$$\begin{pmatrix} \eta_{1,t} - \eta_{2,t} \\ -\eta_{2,t} \end{pmatrix} \Bigg| \varepsilon_t \sim N \left(\begin{bmatrix} (\rho_1 - \rho_2) \varepsilon_t \\ -\rho_2 \varepsilon_t \end{bmatrix}, \begin{bmatrix} 2 - \rho_1^2 - \rho_2^2 & 1 - \rho_2^2 \\ 1 - \rho_2^2 & 1 - \rho_2^2 \end{bmatrix} \right)$$

Appendix A.3: Constructing Forecasts for the Markov-Switching Volatility Feedback Model

In this appendix we describe how to calculate one-step-ahead forecasts for the three-state volatility feedback model with endogenous Markov-switching described in Section 5.2. For the three state model, the one-step-ahead forecast can be expressed as follows:

$$E(r_{t+1}|I_t) = \sum_{j=0}^2 \sum_{i=0}^2 Pr(S_{t+1} = i, S_t = j|I_t) \{ \theta_1 E(\sigma_{S_{t+1}}^2|I_t) + \theta_2 [\sigma_i^2 - E(\sigma_{S_{t+1}}^2|I_t)] + \sigma_i E(\varepsilon_{t+1}|I_t, S_{t+1} = i, S_t = j) \}$$

In the exogenous switching version of the volatility feedback model, $E(\varepsilon_{t+1}|I_t, S_{t+1} = i, S_t = j) = 0$. In the endogenous switching model, these conditional expectations instead equal:

$$\begin{aligned} E(\varepsilon_{t+1}|I_t, S_{t+1} = 0, S_t = j) &= E(\varepsilon_{t+1}|\eta_{1,t+1} < -\gamma_{1,j}, \eta_{2,t+1} < -\gamma_{2,j}) \\ E(\varepsilon_{t+1}|I_t, S_{t+1} = 1, S_t = j) &= E(\varepsilon_{t+1}|\eta_{1,t+1} > -\gamma_{1,j}, \eta_{2,t+1} - \eta_{1,t+1} < (\gamma_{1,j} - \gamma_{2,j})) \\ E(\varepsilon_{t+1}|I_t, S_{t+1} = 2, S_t = j) &= E(\varepsilon_{t+1}|\eta_{1,t+1} - \eta_{2,t+1} < (\gamma_{2,j} - \gamma_{1,j}), \eta_{2,t+1} > -\gamma_{2,j}) \end{aligned}$$

To evaluate these conditional expectations in the endogenous switching case we use simulations. In particular, we generate one million realizations of ε_{t+1} , $\eta_{1,t+1}$, and $\eta_{2,t+1}$ from the multivariate normal distribution in equations (9 - 10), and then calculate the average of ε_{t+1} given the appropriate conditions for each of the conditional expectations. For the two-state endogenous switching volatility feedback model, the relevant conditional expectations are evaluated analogously.

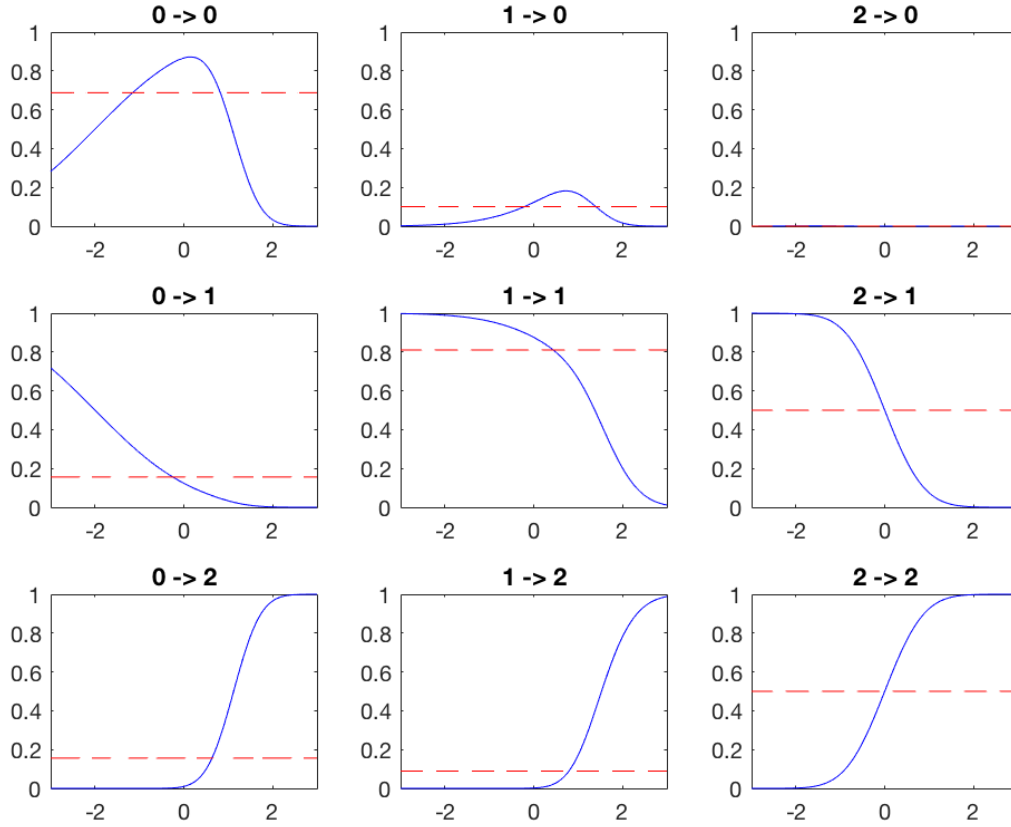
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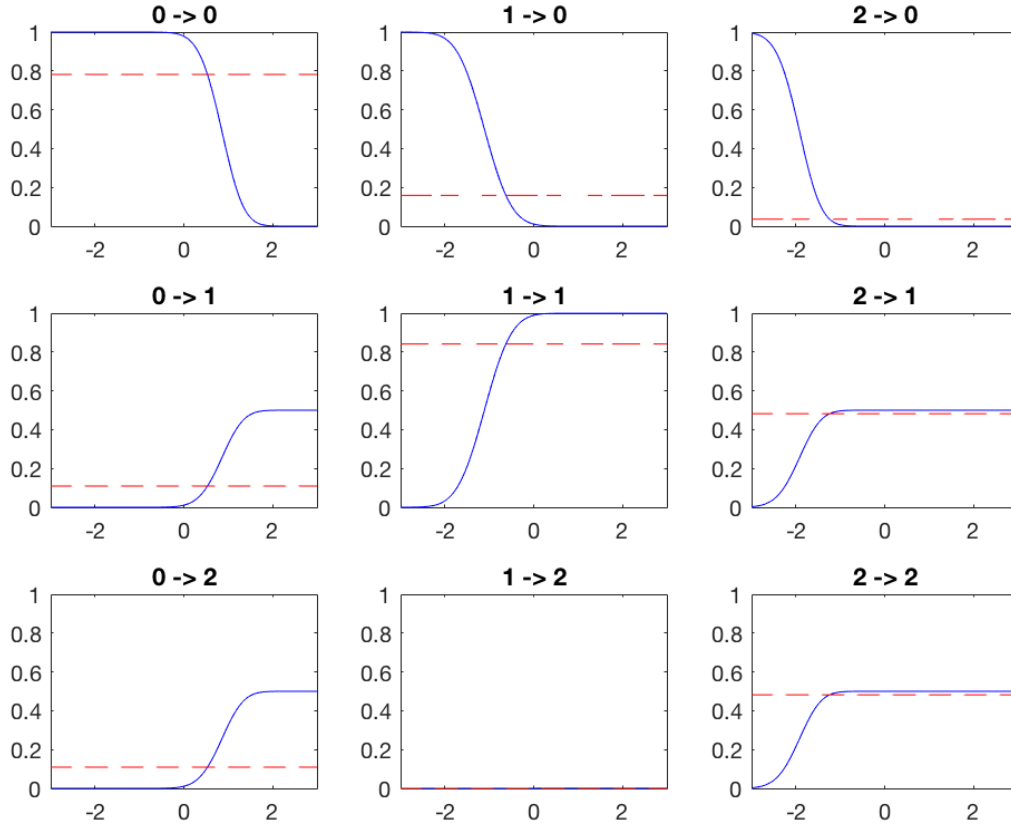
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Figure 1
 $Pr(S_t = i|S_{t-1} = j)$ vs. $Pr(S_t = i|S_{t-1} = j, \varepsilon_t)$
 $\rho_1 = -0.5, \rho_2 = 0.9$



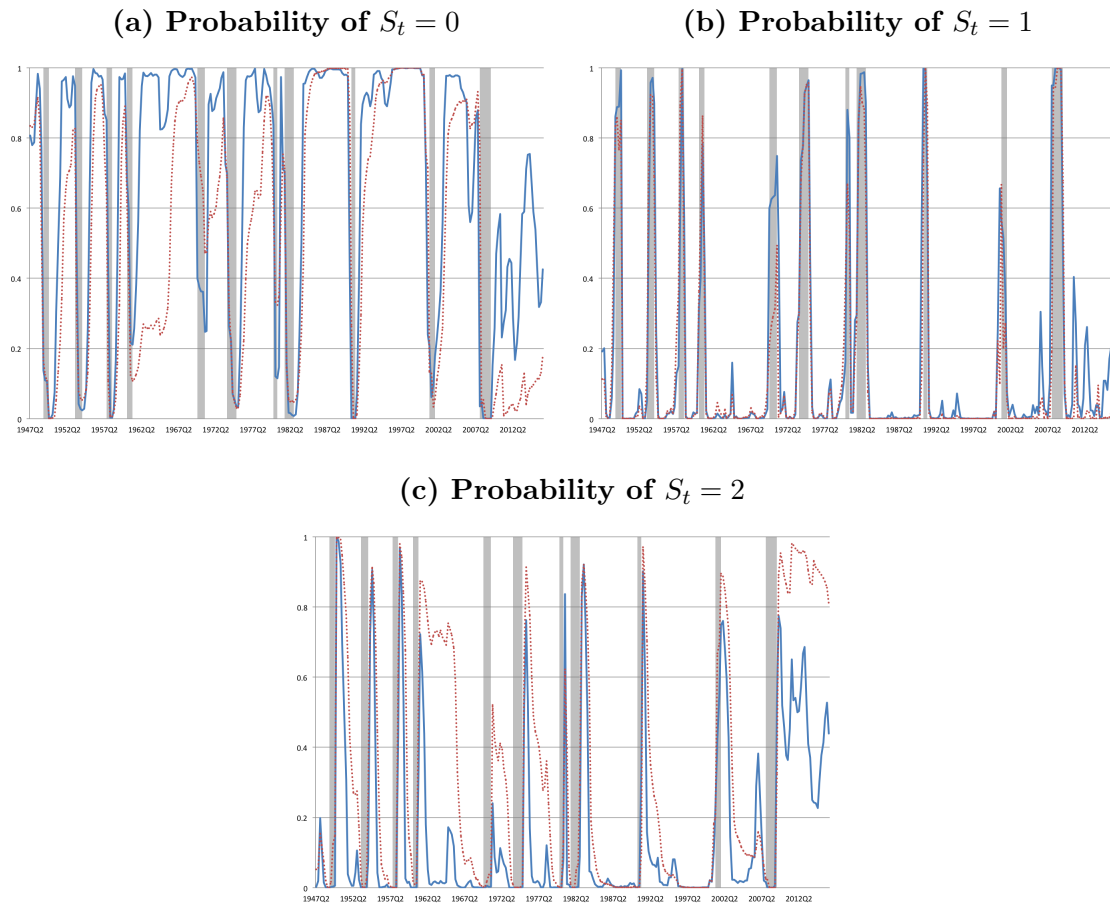
Notes: These graphs show the unconditional transition probability, $Pr(S_t = i|S_{t-1} = j)$ (horizontal dashed line), and the transition probability conditional on the continuous disturbance term in equation (2), $Pr(S_t = i|S_{t-1} = j, \varepsilon_t)$ (solid line). In all panels, $j \rightarrow i$ indicates transitions from state j to state i , and the x-axis measures alternative values of ε_t .

Figure 2
 $Pr(S_t = i|S_{t-1} = j)$ vs. $Pr(S_t = i|S_{t-1} = j, \varepsilon_t)$
 $\rho_1 = 0.9, \rho_2 = 0.9$



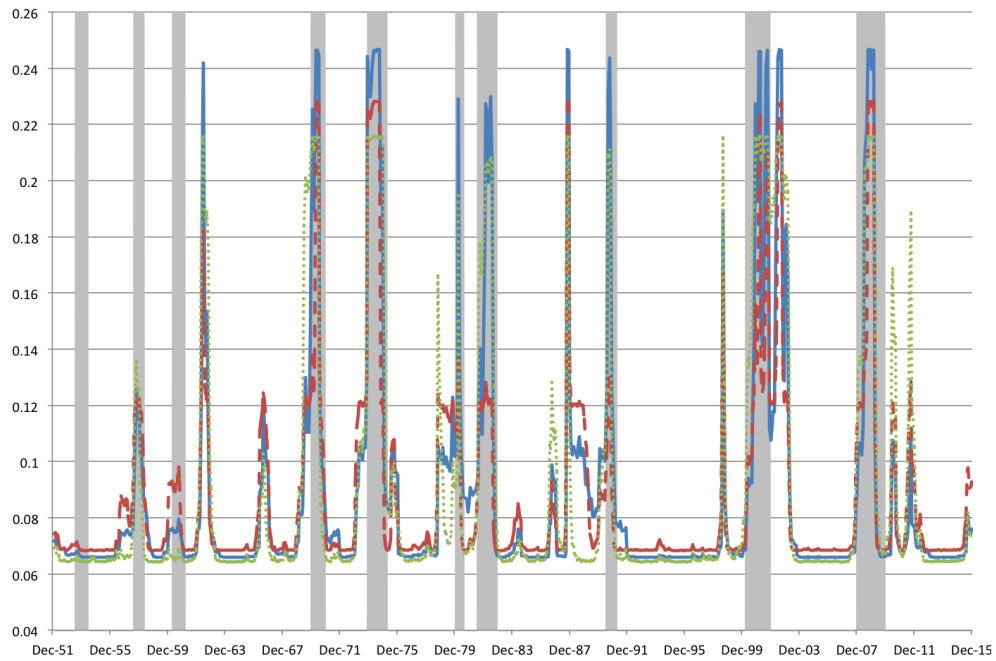
Notes: These graphs show the unconditional transition probability, $Pr(S_t = i|S_{t-1} = j)$ (horizontal dashed line), and the transition probability conditional on the continuous disturbance term in equation (2), $Pr(S_t = i|S_{t-1} = j, \varepsilon_t)$ (solid line). In all panels, $j \rightarrow i$ indicates transitions from state j to state i , and the x-axis measures alternative values of ε_t .

Figure 3
Smoothed State Probabilities for Three Regime Model of Real GDP Growth



Notes: Smoothed probability of mature expansion phase ($S_t = 0$), recession phase ($S_t = 1$), and post-recession recovery phase ($S_t = 2$) from 1947:Q2 to 2016:Q2. Dotted lines denote the regime probability estimated by the exogenous switching model, and solid lines represents the regime probability estimated by the endogenous switching model. NBER recessions are shaded.

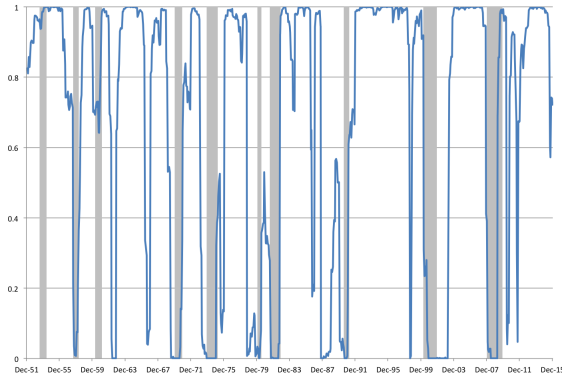
Figure 4
Risk Premium from Alternative Volatility Feedback Models



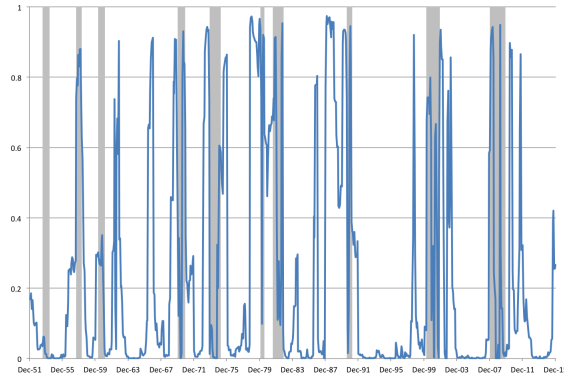
Notes: Risk premium implied by different Markov-switching volatility feedback models. The red dashed line reports the risk premium produced by the exogenous switching model with three states, the green dotted line reports the risk premium produced by the endogenous switching model with two states, and the blue solid line reports the risk premium produced by the endogenous switching model with three states. NBER recessions are shaded.

Figure 5
Smoothed State Probabilities from Three Regime
Volatility Feedback Model with Endogenous Switching

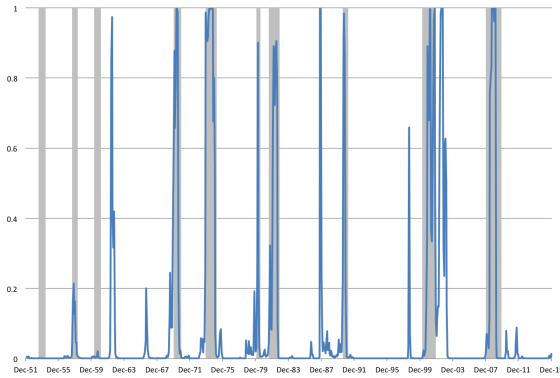
(a) Probability of $S_t = 0$



(b) Probability of $S_t = 1$



(c) Probability of $S_t = 2$



Notes: Smoothed probability of low volatility phase ($S_t = 0$), medium volatility phase ($S_t = 1$), and high volatility phase ($S_t = 2$). NBER recessions are shaded.

Table 1
Monte Carlo Simulation Results
Performance of Misspecified Maximum Likelihood Estimator

	$\rho_1 = \rho_2 = 0.9$			
	$\mu_0 = -1$	$\mu_1 = 0$	$\mu_2 = 1$	$\sigma = 1.0$
<hr/>				
T = 300				
<i>High Persistence</i>	-1.28 (0.11)	0.35 (0.24)	1.58 (0.58)	0.96 (0.01)
<i>Low Persistence</i>	-1.67 (0.46)	0.52 (0.34)	1.64 (0.51)	0.83 (0.04)
T = 500				
<i>High Persistence</i>	-1.27 (0.10)	0.33 (0.20)	1.50 (0.45)	0.97 (0.02)
<i>Low Persistence</i>	-1.66 (0.44)	0.54 (0.33)	1.62 (0.44)	0.84 (0.03)
	$\rho_1 = \rho_2 = 0.5$			
	$\mu_0 = -1$	$\mu_1 = 0$	$\mu_2 = 1$	$\sigma = 1.0$
<hr/>				
T = 300				
<i>High Persistence</i>	-1.20 (0.09)	0.16 (0.13)	1.20 (0.13)	0.97 (0.01)
<i>Low Persistence</i>	-1.42 (0.25)	0.33 (0.29)	1.38 (0.32)	0.92 (0.03)
T = 500				
<i>High Persistence</i>	-1.18 (0.05)	0.12 (0.08)	1.15 (0.07)	0.99 (0.01)
<i>Low Persistence</i>	-1.43 (0.23)	0.27 (0.27)	1.33 (0.26)	0.94 (0.01)
<hr/>				

Notes: This table contains summary results from 1000 Monte Carlo simulations when the true data generating process is given by $y_t = \mu_{S_t} + \sigma \varepsilon_t$ and S_t evolves according to the endogenous switching model detailed in Section 2 with $N=3$ states. “High Persistence” indicates high state persistence, with transition probabilities $p_{00} = p_{11} = p_{22} = 0.9$, while “Low Persistence” indicates transition probabilities $p_{00} = p_{11} = p_{22} = 0.7$. Each cell contains the mean of the 1000 maximum likelihood point estimates for the parameter listed in the column heading, as well as the root mean squared error of the 1000 point estimates from that parameter’s true value (in parentheses). The maximum likelihood estimator is applied to the incorrectly specified model that assumes the state process is exogenous.

Table 2
Monte Carlo Simulation Results
Performance of Correctly Specified Maximum Likelihood Estimator

	$\rho_1 = \rho_2 = 0.9$			
	$\mu_0 = -1$	$\mu_1 = 0$	$\mu_2 = 1$	$\sigma = 1.0$
<hr/>				
T = 300				
<i>High Persistence</i>	-1.02 (0.01)	-0.01 (0.04)	1.02 (0.08)	0.99 (0.01)
<i>Low Persistence</i>	-1.04 (0.02)	-0.02 (0.09)	1.02 (0.11)	0.99 (0.01)
T = 500				
<i>High Persistence</i>	-1.00 (0.01)	-0.01 (0.02)	1.00 (0.02)	1.00 (0.01)
<i>Low Persistence</i>	-1.02 (0.01)	-0.02 (0.06)	1.01 (0.05)	1.00 (0.01)
	$\rho_1 = \rho_2 = 0.5$			
	$\mu_0 = -1$	$\mu_1 = 0$	$\mu_2 = 1$	$\sigma = 1.0$
<hr/>				
T = 300				
<i>High Persistence</i>	-1.02 (0.03)	0.04 (0.09)	1.05 (0.08)	1.00 (0.01)
<i>Low Persistence</i>	-1.05 (0.08)	0.05 (0.16)	1.11 (0.32)	1.01 (0.01)
T = 500				
<i>High Persistence</i>	-1.02 (0.02)	0.01 (0.06)	1.04 (0.06)	1.00 (0.01)
<i>Low Persistence</i>	-1.03 (0.05)	0.03 (0.14)	1.02 (0.13)	1.01 (0.01)

Notes: This table contains summary results from 1000 Monte Carlo simulations when the true data generating process is given by $y_t = \mu_{S_t} + \sigma \varepsilon_t$ and S_t evolves according to the endogenous switching model detailed in Section 2 with $N=3$ states. “High Persistence” indicates high state persistence, with transition probabilities $p_{00} = p_{11} = p_{22} = 0.9$, while “Low Persistence” indicates transition probabilities $p_{00} = p_{11} = p_{22} = 0.7$. Each cell contains the mean of the 1000 maximum likelihood point estimates for the parameter listed in the column heading, as well as the root mean squared error of the 1000 point estimates from that parameter’s true value (in parentheses). The maximum likelihood estimator is applied to the correctly specified model that assumes the state process is endogenous.

Table 3
Monte Carlo Simulation Results
Size and Size-Adjusted Power of Tests of $\rho_1 = \rho_2 = 0$

	Size		Power		Power	
	$\rho_1 = \rho_2 = 0$		$\rho_1 = \rho_2 = 0.5$		$\rho_1 = \rho_2 = 0.9$	
	Wald	LR	Wald	LR	Wald	LR
$T = 300$						
High Persistence	9.5	7.5	47.3	73.3	84.2	98.7
Low Persistence	10.3	8.1	42.2	68.8	81.3	97.3
$T = 500$						
High Persistence	7.2	5.4	68.1	84.5	90.1	100
Low Persistence	7.7	5.7	64.2	80.2	88.7	100

Notes: Each cell of the table contains the percentage of 1000 Monte Carlo simulations for which the Wald test or likelihood ratio (LR) test rejected the null hypothesis that $\rho_1 = \rho_2 = 0$ at the 5% significance level. For columns labeled “Size”, critical values are based on the asymptotic distribution of the test-statistic. For columns labeled “Power”, size adjusted critical values are calculated from 1000 simulated test statistics from the corresponding Monte Carlo experiment in which $\rho_1 = \rho_2 = 0$. The data generating process used to simulate the Monte Carlo samples is given by $y_t = \mu_{S_t} + \sigma \varepsilon_t$, where S_t evolves according to the endogenous switching model detailed in Section 2 with $N=3$ states and ρ_1 and ρ_2 given by the column headings.

Table 4
Monte Carlo Simulation Results
Selection of Number of States using Markov-Switching Information Criteria

	$\rho_1 = \rho_2 = 0.9$		
	$N = 2$	$N = 3$	$N = 4$
<hr/>			
$T = 300$			
<i>High Persistence</i>	0%	99.4%	0.6%
<i>Low Persistence</i>	0%	97.3%	2.7%
$T = 500$			
<i>High Persistence</i>	0%	99.6%	0.4%
<i>Low Persistence</i>	0%	98.9%	1.1%
	$\rho_1 = \rho_2 = 0.5$		
	$N = 2$	$N = 3$	$N = 4$
<hr/>			
$T = 300$			
<i>High Persistence</i>	0%	99.1%	0.9%
<i>Low Persistence</i>	0%	98.1%	1.9%
$T = 500$			
<i>High Persistence</i>	0%	99.5%	0.5%
<i>Low Persistence</i>	0%	99.3%	0.7%
<hr/>			

Notes: Each cell of the table contains the percentage of 1000 Monte Carlo simulations for which the Markov-Switching Information Criteria (MSC) of Smith et al. (2006) selected the indicated number of states (N). The data generating process used to simulate the Monte Carlo samples is given by $y_t = \mu_{S_t} + \sigma \varepsilon_t$, where S_t evolves according to the endogenous switching model detailed in Section 2 with $N=3$ states and ρ_1 and ρ_2 given by the column headings. “High Persistence” indicates high state persistence, with transition probabilities $p_{00} = p_{11} = p_{22} = 0.9$, while “Low Persistence” indicates transition probabilities $p_{00} = p_{11} = p_{22} = 0.7$.

Table 5
U.S. Real GDP Growth Rate in Quarters
Following Post-War U.S. Recessions

Quarters After Recession	Average Growth	Observations
1	6.45	11
2	6.32	11
3	5.50	11
4	5.79	10
5	4.14	10
6	4.29	10
7	3.66	10
8	3.48	9
Full Sample	3.10	278

Notes: Average growth rates are measured as annualized percentages. The sample period is 1947:Q1 to 2016:Q2. For four quarters and longer, one observation is lost due to the termination of the expansion following the 1980 recession. For eight quarters, another observation is lost due to the termination of the expansion following the 1957-1958 recession.

Table 6
Regime Switching Model for Real GDP Growth Rate

	Exogenous Switching		Endogenous Switching	
μ_0	0.87	(0.06)	0.93	(0.11)
μ_1	-0.48	(0.18)	-0.46	(0.18)
$\mu_{2,1}$	1.62	(0.45)	2.37	(0.42)
$\mu_{2,2}$	0.54	(0.15)	0.34	(0.13)
p_{00}	0.93	(0.03)	0.93	(0.03)
p_{11}	0.71	(0.09)	0.69	(0.10)
p_{22}	0.88	(0.05)	0.72	(0.09)
σ_1	0.92	(0.07)	0.93	(0.08)
σ_2	0.46	(0.03)	0.50	(0.05)
$\rho_{1,1}$			-0.30	(0.29)
$\rho_{1,2}$			0.85	(0.16)
$\rho_{2,1}$			-0.79	(0.19)
$\rho_{2,2}$			0.80	(0.19)
Log Likelihood	-323.64		-318.78	
	<i>Q</i> -statistic	p-value	<i>Q</i> -statistic	p-value
$Q(k = 1)$	4.07	0.04	0.29	0.58
$Q(k = 2)$	5.48	0.02	1.33	0.52
$Q(k = 4)$	7.35	0.12	5.35	0.25

Notes: This table reports maximum likelihood estimates of the three state switching mean model of U.S. real GDP growth given in equation (22). The sample period is 1947:Q1 to 2016:Q2. Standard errors, reported in parentheses, are based on second derivatives of the log-likelihood function in all cases. $Q(k)$ stands for the Ljung-Box test statistic for serial correlation in the standardized disturbance term calculated by smoothed probabilities up to k lags.

Table 7
Volatility Feedback Model with Three Volatility States

	Exogenous Switching		Endogenous Switching	
θ_1	0.50	(0.11)	0.47	(0.11)
θ_2	-5.94	(1.67)	-3.36	(1.03)
p_{00}	0.96	(0.01)	0.97	(0.01)
p_{10}	0.04	(0.01)	0.03	(0.01)
p_{01}	0.09	(0.02)	0.07	(0.01)
p_{11}	0.85	(0.03)	0.84	(0.03)
p_{12}	0.30	(0.08)	0.31	(0.05)
p_{22}	0.67	(0.11)	0.69	(0.05)
σ_0	0.37	(0.01)	0.38	(0.01)
σ_1	0.49	(0.03)	0.46	(0.02)
σ_2	0.68	(0.07)	0.73	(0.08)
ρ_1			-0.63	(0.15)
ρ_2			0.42	(0.30)
Log Likelihood	-513.90		-510.51	
	<i>Q</i> -statistic	p-value	<i>Q</i> -statistic	p-value
$Q(k = 1)$	0.40	0.53	0.14	0.71
$Q(k = 2)$	0.69	0.71	0.38	0.83
$Q(k = 4)$	3.15	0.54	0.90	0.92
	<i>ARCH</i> -statistic	p-value	<i>ARCH</i> -statistic	p-value
$ARCH(k = 1)$	4.42	0.04	0.73	0.39
$ARCH(k = 2)$	6.85	0.03	3.47	0.18
$ARCH(k = 4)$	11.11	0.03	6.53	0.16

Notes: This table reports maximum likelihood estimates of the volatility feedback model in equation (23) with three volatility states, estimated assuming both exogenous and endogenous Markov switching. The sample period is 1952:M1 to 2015:M12. Standard errors, reported in parentheses, are based on second derivatives of the log-likelihood function in all cases. $Q(k)$ stands for the Ljung-Box test statistic for serial correlation in the standardized disturbance term calculated by smoothed probabilities up to k lags. $ARCH(k)$ stands for the ARCH-LM test statistic for serial correlation in the squared standardized disturbance term calculated by smoothed probabilities up to k lags.

Table 8
RMSE for Pseudo Out-of-Sample Forecasts of Equity Returns
from Volatility Feedback Model

	Exogenous Switching	Endogenous Switching
Two States	0.556	0.552
Three States	0.539	0.535

Notes: This table reports the root mean squared errors (RMSE) for pseudo out-of-sample forecasts of excess equity returns from alternative versions of a Markov-switching volatility feedback model. Versions of the model are estimated with either two or three volatility states, and assumptions of either exogenous or endogenous Markov switching. The out-of-sample period is January 2000 to December 2015. Details of the volatility feedback model are described in Section 5.2.