Unpublished Appendix for:

"Estimation of Markov Regime-Switching Regression Models with Endogenous Switching"

Chang-Jin Kim, Jeremy Piger and Richard Startz

Derivation of equations (2.9) and (3.3):

We proceed by generalizing the derivation of the univariate skew-normal density function given in Arnold and Beaver (2002). The random variables described in equation (2.5) can be written as:

$$\begin{bmatrix} \varepsilon_t \\ \eta_t \end{bmatrix} = A \begin{bmatrix} \varepsilon_t \\ \omega_t \end{bmatrix} , \tag{A.1}$$

where $\omega_t \sim i.i.d.N(0,1)$, and $A = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix}$ is the Cholesky decomposition of Σ, so that $AA^{'} = \Sigma$. From (A.1):

$$\eta_t = \rho\varepsilon_t + \sqrt{1-\rho^2}\,\omega_t . \tag{A.2}$$

We can then write, suppressing Ω_t, ξ_{t-1}, and θ from the conditioning set for convenience:

$$f(y_t \mid S_t = i, S_{t-1} = j)$$

$$= f\left(y_t \mid c_{i-1,j,t} \le \eta_t < c_{i,j,t}\right)$$

$$= f\left(y_t \mid \frac{c_{i-1,j,t} - \rho\varepsilon_t}{\sqrt{1-\rho^2}} \le \omega_t < \frac{c_{i,j,t} - \rho\varepsilon_t}{\sqrt{1-\rho^2}} \right), \tag{A.3}$$

where $c_{i-1,j,t}$ and $c_{i,j,t}$ are defined in Section 3. Consider the cumulative probability distribution function:

$$\Pr\left(y_t < g \mid \frac{c_{i-1,j,t} - \rho\varepsilon_t}{\sqrt{1-\rho^2}} \le \omega_t < \frac{c_{i,j,t} - \rho\varepsilon_t}{\sqrt{1-\rho^2}} \right) \tag{A.4}$$

$$= \frac{\Pr\left(y_t < g, \frac{c_{i-1,j,t} - \rho\varepsilon_t}{\sqrt{1-\rho^2}} \le \omega_t < \frac{c_{i,j,t} - \rho\varepsilon_t}{\sqrt{1-\rho^2}} \right)}{\Pr\left(\frac{c_{i-1,j,t} - \rho\varepsilon_t}{\sqrt{1-\rho^2}} \le \omega_t < \frac{c_{i,j,t} - \rho\varepsilon_t}{\sqrt{1-\rho^2}} \right)}$$

The denominator of (A.4) is:

$$\Pr\left(\frac{c_{i-1,j,t} - \rho\varepsilon_t}{\sqrt{1-\rho^2}} \le \omega_t < \frac{c_{i,j,t} - \rho\varepsilon_t}{\sqrt{1-\rho^2}} \right) = p_{ij}(z_t) = \Phi\left(c_{i,j,t}\right) - \Phi\left(c_{i-1,j,t}\right). \tag{A.5}$$

The numerator of (A.4) is:

$$\Pr\left(y_t < g, \frac{c_{i-1,j,t} - \rho\varepsilon_t}{\sqrt{1-\rho^2}} \le \omega_t < \frac{c_{i,j,t} - \rho\varepsilon_t}{\sqrt{1-\rho^2}} \right)$$

$$= \int_{-\infty}^{g} \int_{\left(c_{i-1,j,t} - \rho\varepsilon_t\right)/\sqrt{1-\rho^2}}^{\left(c_{i,j,t} - \rho\varepsilon_t\right)/\sqrt{1-\rho^2}} f(y_t, \omega_t) d\omega d\,\varepsilon$$

$$= \int_{-\infty}^{g} \int_{\left(c_{i-1,j,t} - \rho\varepsilon_t\right)/\sqrt{1-\rho^2}}^{\left(c_{i,j,t} - \rho\varepsilon_t\right)/\sqrt{1-\rho^2}} \frac{1}{\sigma_i} \phi\left(\frac{y_t - x_t^{'}\beta_i}{\sigma_i} \right) f\left(\omega_t\right) d\omega d\varepsilon$$

$$= \int_{-\infty}^{g} \frac{1}{\sigma_i} \phi\left(\frac{y_t - x_t^{'}\beta_i}{\sigma_i}\right)\left(\Phi\left(\frac{c_{i,j,t} - \rho\left(\frac{y_t - x_t^{'}\beta_i}{\sigma_i}\right)}{\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{c_{i-1,j,t} - \rho\left(\frac{y_t - x_t^{'}\beta_i}{\sigma_i}\right)}{\sqrt{1-\rho^2}}\right)\right) d\varepsilon . \qquad \text{(A.6)}$$

Combining (A.5)-(A.6) and differentiating with respect to g yields:

$$f(y_t \mid S_t = i, S_{t-1} = j) \qquad \text{(A.7)}$$

$$= \frac{\phi\left(\frac{y_t - x_t^{'}\beta_i}{\sigma_i}\right)\left(\Phi\left(\frac{c_{i,j,t} - \rho\left(\frac{y_t - x_t^{'}\beta_i}{\sigma_i}\right)}{\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{c_{i-1,j,t} - \rho\left(\frac{y_t - x_t^{'}\beta_i}{\sigma_i}\right)}{\sqrt{1-\rho^2}}\right)\right)}{\sigma_i p_{ij}(z_t)},$$

which is the density function in equation (3.3). When $N = 2$ we have:

$$f\left(y_t \mid S_t = 1, S_{t-1} = j\right) = \frac{\phi\left(\frac{y_t - x_t^{'}\beta_1}{\sigma_1}\right)\Phi\left(\frac{a_{1,j} + z_t^{'}b_{1,j} - \rho\left(\frac{y_t - x_t^{'}\beta_1}{\sigma_1}\right)}{\sqrt{1-\rho^2}}\right)}{\sigma_1 p_{1j}(z_t)}$$

$$f\left(y_t \mid S_t = 2, S_{t-1} = j\right) = \frac{\phi\left(\frac{y_t - x_t^{'}\beta_2}{\sigma_2}\right)\Phi\left(\frac{-\left(a_{1,j} + z_t^{'}b_{1,j}\right) + \rho\left(\frac{y_t - x_t^{'}\beta_2}{\sigma_2}\right)}{\sqrt{1-\rho^2}}\right)}{\sigma_2 p_{2j}(z_t)},$$

which, upon renaming $a_{1,j} = a_j$ and $b_{1,j} = b_j$, is the density function in equation (2.9).